Gauge momentum operators for the Calogero-Sutherland model with anti-periodic boundary condition

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The integrability of a classical Calogero systems with anti-periodic boundary condition is studied. This system is equivalent to the periodic model in the presence of a magnetic field. Gauge momentum operators for the anti-periodic Calogero system are constructed. These operators are hermitian and simultaneously diagonalizable with the Hamiltonian. A general scheme for constructing such momentum operators for trigonometric and hyperbolic Calogero-Sutherland model is proposed. The scheme is applicable for both periodic and anti-periodic boundary conditions. The existence of these momentum operators ensures the integrability of the system. The interaction parameter λ is restricted to a certain subset of real numbers. This restriction is in fact essential for the construction of the hermitian gauge momentum operators.

Keywords: exact results, low dimensional quantum mechanics and quantum field theory, algebraic structure of integrable model

I. INTRODUCTION

The study of integrable and solvable quantum many-body problem has become an active field of research with many applications in various branches of physics and mathematics. One dimensional models with two-body inverse square long-range interaction are of great

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physical interest as they are extensively used to estimate the physical feature of several condensed matter systems, e.g., quantum hall effect[1], Luttinger liquid[2]. They also have deep connections with Yang-Mills theories[3,4], soliton theory[5], random matrix models[6], multivariable orthogonal polynomials[7], quantum gravity and black holes[8]. In systems of physical interest, these one-dimensional models with appropriate inverse square long-range interaction are exactly solvable due to the highly restrictive spatial degrees of freedom. The spatial restriction however, introduces large quantum fluctuations resulting in failure of the mean field approach which works well in higher dimensional systems.

Several integrable models in one dimension have been proposed and constructed using the inverse square potential. The Calogero-Sutherland model (CSM)[9,10] is one such model with applications in several physical systems. This model provides a clear explanation of fractional exchange and exclusion statistics. In most other one-dimensional models, the definition of fractional exchange statistics is rather obscure and incomplete. For spinless CSM, the fractional exchange statistics can be formulated in the language of first quantization by using the one dimensional analogue of Chern-Symon Gauge theories[11].

The CSM is also useful in the study of the fractional exclusion statistics based on the generalization of the exclusion principle[12]. The exclusion statistics is usually interpreted in terms of real, pseudo and quasi momenta which describe the particle and hole type excitations of one dimensional systems[13]. In Calogero systems it is observed that two neighboring pseudo-momenta are always separated by a number that depends on a statistical parameter present in the Hamiltonian. The study of such particle and hole type excitations is important for constructing the basic thermodynamic functions of a system. In addition, the CSM Hamiltonian can be used to construct an effective low energy model for anyons following Luttinger liquid theory[2]. For integer values of interaction parameter, this type of one dimensional anyon system is equivalent to a coupled system of left and right moving edge-states of fractional quantum Hall effect[14].

The N-particle Hamiltonian of a general Calogero model represents a system of spinless non-relativistic particles interacting through a two-body potential and may be written as,

$$H_N = \sum_{j=1}^{N} {\partial_j}^2 - \lambda(\lambda-1)\sum_{\substack{j,k \\ j\neq k}} U(x_{jk}) = \sum_{j=1}^{N} {\partial_j}^2 - \lambda(\lambda-1){\sum_{j,k}}' U(x_{jk}) \tag{1}$$

where, prime over the summation sign implies that the terms with $j = k$ are omitted.

The two-body long-range potential is represented by $U(x_{jk})$, where $x_{jk} = x_j - x_k$ is the distance between particles at j-th and k-th sites and λ is a dimensionless interaction parameter. The two-body potential $U(x)$ is an even function and under periodic boundary condition has a general expression in terms of Weierstrass elliptic function[15]. This may be further reduced to the trigonometric, hyperbolic and rational function by means of a limiting procedure[16]. The original version of the Calogero model assumed a two-body inverse square potential. The model was shown to be integrable by Calogero and Perelomov[17,18] using quantum Lax formulation and its explicit integration was performed by Krichever[19]. The existence of a complete set of mutually commuting momentum operators that commute with the Hamiltonian as well, also establishes the integrability of this model. The CSM with periodic boundary condition has been well studied in literature[7,10,20,21,22]. During the past decades the CSMs (both classical and spin system) have been actively explored in a variety of ways including the exchange operator formalism (EOF)[23], the Dunkl operator approach[20], reduction by discrete symmetries[24] and construction of Lax-pair[25]. The operators used in EOF or quantum Lax formulation are associated with certain types of root systems. The Calogero type models may be obtained from the projection of free motion on a higher dimensional manifold.

This article investigates the Calogero systems for trigonometric and hyperbolic types of long-range interactions with anti-periodic boundary conditions[26]. Such systems are equivalent to a periodic model in the presence of an external magnetic field. The trigonometric form of the Hamiltonian is obtained by mapping the one dimensional chain of particles on a circular ring. The hyperbolic version of the model is derived by expressing the potential in terms of Weierstrass $\wp$-function[27]. The integrability is established by constructing gauge momentum operators that are hermitian and are simultaneously diagonalizable with the Hamiltonian. A general scheme for constructing such momentum operators is proposed, for trigonometric and hyperbolic types of Calogero systems with both periodic and anti-periodic boundary conditions. The existence of these commuting momentum operators ensures the integrability of the models under consideration. It is also shown in this article, that for periodic boundary condition the gauge momentum operator exists for any real values of λ whereas for anti-periodic case there is a restriction on the range of allowed values of λ. Moreover, for bosonic ($\lambda = 0$) and fermionic ($\lambda = 1$) limits, both periodic and anti-periodic models have identical spectra but this similarity is absent for any other value of λ.

II. THE CALOGERO-SUTHERLAND MODEL WITH TRIGONOMETRIC TYPE INTERACTION

A. Trigonometric CSM with periodic boundary condition

Let us first consider the general CSM; a one dimensional chain of classical particles with inverse square long-range interaction. The topological representation of this one dimensional chain is simply a circular ring. In the absence of a magnetic field, a particle transported adiabatically around the ring an integral number of times, does not take up any phase factor, and hence the eigenfunctions retain their initial form. Thus, the pairwise interaction summed over all possible pairs, around a circle of circumference L, an infinite number of times ($\nu \to \infty$) is given as,

$$\lim_{\nu \to \infty} \sum_{n=-\nu}^{\nu} \frac{1}{(x+nL)^2} = \frac{1}{[d(x)]^2}, \tag{2}$$

where x is the distance along the arc of the circle, between the particles at the j-th and k-th sites, and $d(x_{jk})$ is the the chord distance between them. This chord length is given by, (See Fig. 1)

$$d(x) = \frac{L}{\pi} \sin\left(\frac{\pi x}{L}\right). \tag{3}$$

Hence,

$$U(x) = \frac{\pi^2}{L^2} \frac{1}{\sin^2\left(\frac{\pi x}{L}\right)}$$

The Hamiltonian then is given by,

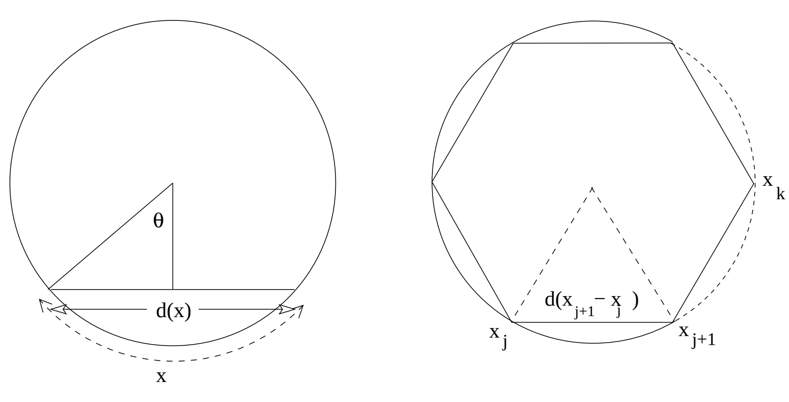

FIG. 1: Interparticle distances $d(x)$ and $d(x_{j+1} - x_j)$ for particles on a circular chain

$$H_N = \sum_{j=1}^{N} {\partial_j}^2 - \lambda(\lambda-1)\frac{\pi^2}{L^2}{\sum_{j,k}}' \frac{1}{\sin^2\left(\frac{\pi}{L}x_{jk}\right)}. \tag{4}$$

Using standard trigonometric identity, Eq.(4) becomes,

$$H_N = \sum_{j=1}^{N} {\partial_j}^2 - \frac{\pi^2}{4L^2}\lambda(\lambda-1){\sum_{j,k}}' \left(\frac{1}{\sin^2\left(\frac{\pi}{2L}x_{jk}\right)} + \frac{1}{\cos^2\left(\frac{\pi}{2L}x_{jk}\right)}\right). \tag{5}$$

Making a change of variable $(\pi/2L)x_j \to x_j$ and rescaling the Hamiltonian as $(4L^2/\pi^2)H \to H$, we get the trigonometric Hamiltonian with periodic boundary condition as follows

$$H_N^{t+} = \sum_{j=1}^{N} {\partial_j}^2 - \lambda(\lambda-1){\sum_{j,k}}' \left(\frac{1}{\sin^2(x_{jk})} + \frac{1}{\cos^2(x_{jk})}\right). \tag{6}$$

B. Trigonometric CSM with anti-periodic boundary condition

The two-body interaction term in CSM Hamiltonian takes a different form when the boundary condition is changed to anti-periodic. The change in boundary condition signifies a change of certain symmetry consideration in the underlying algebraic structure of the model. A general twisted boundary condition arises when a magnetic field is applied transverse to the one dimensional ring considered above. Then, if a particle is transported adiabatically around the entire system n number of times, it picks up a net phase $\exp(in\phi)$.
The total pairwise interaction summed over all possible pairs an infinite number of times around a circle of circumference L is given by,

$$\sum_{n=-\infty}^{+\infty} \frac{\exp(i\phi n)}{(x+nL)^2}. \tag{7}$$

The above summation can be evaluated if we make the choice,

$$\begin{aligned} &\phi = 2\pi p/q, \quad \text{with} \quad p,\, q \quad \text{relative primes; and} \\ &n = jq + k, \quad \text{with} \quad j,\, k \quad \text{integers such that} \quad -\infty < j < +\infty, \quad \text{and} \quad 0 \le k \le (q-1). \end{aligned} \tag{8}$$

Using the above, the interaction term becomes,

$$\sum_{n=-\infty}^{+\infty} \frac{\exp(i\phi n)}{(x+nL)^2} = \sum_{k=0}^{q-1}\sum_{j=-\infty}^{+\infty} \frac{\exp(i2\pi pj)\exp(i2\pi pk/q)}{\left[(x+kL)+(qL)j\right]^2} = \sum_{k=0}^{q-1} \frac{\exp(i2\pi pk/q)}{\left[(qL/\pi)\sin[(\pi(x+kL))/qL]\right]^2} \tag{9}$$

The last expression represents an interaction when the system is subjected to a general twisted boundary condition. The model can be viewed as a system of interacting particles residing on a circle with circumference qL. For $p/q = 1/2$ this corresponds to an anti-periodic boundary condition[26]. In this case the sum in Eq.(9) may be written as

$$\sum_{k=0}^{1} \frac{\exp\left(i\pi k\right)}{[(2L/\pi)\sin[(\pi(x+kL))/2L]]^2} = \frac{1}{[(2L/\pi)\sin[(\pi x)/2L]]^2} - \frac{1}{[(2L/\pi)\sin[(\pi(x+L))/2L]]^2}$$
$$= \frac{\pi^2}{4L^2\sin^2[(\pi x)/2L]} - \frac{\pi^2}{4L^2\cos^2[(\pi x)/2L]}. \tag{10}$$

The above expression represents the potential term for the Calogero system with anti-periodic boundary condition. Making a change of variable $(\pi/2L)x_j \rightarrow x_j$ and rescaling the Hamiltonian as $(4L^2/\pi^2)H \rightarrow H$, the trigonometric Hamiltonian with anti-periodic boundary condition, denoted by H_N^{t-}, is written as

$$H_N^{t-} = \sum_{j=1}^{N} {\partial_j}^2 - \lambda(\lambda-1)\sum_{j,k}{}' \left(\frac{1}{\sin^2\left(x_{jk}\right)} - \frac{1}{\cos^2\left(x_{jk}\right)} \right). \tag{11}$$

The two Hamiltonians in Eq.(6) and Eq.(11) may be expressed in a compact form as,

$$H_N^{t\pm} = \sum_{j=1}^{N} {\partial_j}^2 - \lambda(\lambda-1)\sum_{j,k}{}' [V(x_{jk}) \pm R(x_{jk})], \tag{12}$$

where, the upper sign is for the periodic and the lower sign is for the anti-periodic case. $V(x)$ and $R(x)$ are even functions of x.

C. Construction of commuting gauge momentum operators

In order to construct the gauge momentum operators we introduce the exchange operator Λ_{jk} which preserves the function space under exchange of coordinates of the particles. The exchange operator Λ_{jk} has the following properties,

1. $\Lambda_{jk} f(x_1, .., x_j, .., x_k, .., x_N) = f(x_1, .., x_k, .., x_j, .., x_N)$.

2. $\Lambda_{jk} = \Lambda_{kj}$.

3. $\Lambda_{jk}^2 = 1$.

4. $\Lambda_{ij}\Lambda_{jk} = \Lambda_{ik}\Lambda_{ij} = \Lambda_{jk}\Lambda_{ik}$.

5. $\Lambda_{ij}\Lambda_{kl} = \Lambda_{kl}\Lambda_{ij}$.

6. $\Lambda_{jk} x_k = x_j$.

Let us define

$$v(x) = \frac{1}{2}\frac{d}{dx}\ln V(x), \qquad\qquad r(x) = \frac{1}{2}\frac{d}{dx}\ln R(x). \tag{13}$$

It can be easily shown that $r(x) \cdot v(x) = -1$. From the definition it is clear that $v(x)$ and $r(x)$ are odd functions of x. Using the exchange operator Λ_{ij}, we define the gauge momentum operators $\{d_j | j = 1 \ldots N\}$ in terms of the functions $v(x)$ and $r(x)$ as

$$d_j = \partial_j + \mu_1(\lambda) \sum_{\substack{k \\ k\neq j}} v(x_{jk})\Lambda_{jk} + \mu_2(\lambda) \sum_{\substack{k \\ k\neq j}} r(x_{jk})\Lambda_{jk} \tag{14}$$

$\mu_1(\lambda)$and $\mu_2(\lambda)$ being real functions of λ. To ensure integrability, we require the momentum operators to satisfy the following relations,

$$[\Lambda_{ij}, d_k] = 0. \tag{15}$$

$$\sum_{j=1}^{N} d_j^2 = H_N^{t\pm} + \text{constant}. \tag{16}$$

$$[d_j, d_k] = 0. \tag{17}$$

$$[d_j, H_N^{t\pm}] = 0. \tag{18}$$

This implies that $r(x)$ and $v(x)$ must obey the following restrictions

$$\frac{d}{dx}v(x) = V(x)\ , \qquad\qquad v^2(x) = V(x) + \text{constant} \tag{19}$$

$$\frac{d}{dx}r(x) = R(x)\ , \qquad\qquad r^2(x) = R(x) + \text{constant} \tag{20}$$

Making use of the properties of the momentum operators (Eq.(15) -Eq.(18)) and Eq.(19), the momentum operators $\{d_j | j = 1 \ldots N\}$ in Eq.(14) can be written as,

$$d_j = \partial_j - \mu_1(\lambda) \sum_{\substack{k \\ k\neq j}} \cot(x_{jk})\Lambda_{jk} + \mu_2(\lambda) \sum_{\substack{k \\ k\neq j}} \tan(x_{jk})\Lambda_{jk}. \tag{21}$$

For the system with periodic boundary condition, Eq.(16) is true for $\mu_1(\lambda) = \lambda, 1 - \lambda$ and $\mu_2(\lambda) = \lambda, 1 - \lambda$ whereas for anti-periodic boundary condition, the same form of momentum operators demand $\mu_1(\lambda) = \lambda, 1 - \lambda$ and $\mu_2(\lambda) = \frac{1}{2}[1 \pm \sqrt{1 + 4\lambda - 4\lambda^2}]$. From the above values of $\mu_1(\lambda)$ and $\mu_2(\lambda)$ it is observed that for bosonic ($\lambda = 0$) and fermionic ($\lambda = 1$)

limits, the spectra of periodic and anti-periodic models are identical. There does not exist any other λ, for which such an identical spectrum is obtained for periodic and anti-periodic models. This can be readily checked by putting $\lambda = 0, 1$ in the respective Hamiltonians.

It is further noted that the spectrum for anti-periodic case is real for a restricted range of values of λ, i.e., $\frac{1}{2} - \frac{1}{\sqrt{2}} \leq \lambda \leq \frac{1}{2} + \frac{1}{\sqrt{2}}$ unlike its periodic counterpart.

III. THE CALOGERO-SUTHERLAND MODEL WITH HYPERBOLIC TYPE INTERACTION

The hyperbolic form of the CSM may be obtained by taking the limit of the Weierstrass $\wp$-function which is a doubly periodic even elliptic function. It is an analytic function except at points which are double poles congruent to the vertices of the period parallelogram. Let us consider the Weierstrass σ-function given as[27]

$$\sigma(x) = \frac{2P_1}{\pi} \exp\left(\frac{\eta_1 x^2}{2P_1}\right) \sin\left(\frac{\pi x}{2P_1}\right) \prod_{n=1}^{\infty} \left(\frac{1 - 2q^{2n}\cos\left(\frac{\pi x}{P_1}\right) + q^{4n}}{(1-q^{2n})^2}\right), \tag{22}$$

where, P_1, P_2 are the half-period magnitudes, $q = \exp(i\pi P_2/P_1)$ and $\eta_1 = \frac{d}{dz} \ln \sigma(z)|_{P_1}$. The $\wp$-function with period $2P_1$ and $2P_2$ is given by

$$\wp(x|2P_1, 2P_2) = -\frac{d^2}{dx^2} \ln \sigma(x). \tag{23}$$

A. Hyperbolic extension of the CSM with periodic and anti-periodic boundary conditions

The CSM Hamiltonian with elliptic type interaction under periodic and anti-periodic boundary condition is written as

$$H_N^e = \sum_{j=1}^{N} {\partial_j}^2 - \lambda(\lambda-1){\sum_{j,k}}' [\wp(x_{jk}|2P_1, 2P_2) \pm \wp(x_{jk} + P_1|2P_1, 2P_2)]. \tag{24}$$

The plus and minus sign represent the periodic and anti-periodic cases respectively. In view of Eq.(23) and Eq.(22)

$$\begin{aligned} \wp(x|2P_1, 2P_2)|_{q\to 0} &= \frac{\pi^2}{4P_1^2} \csc^2\left(\frac{\pi x}{2P_1}\right) + C_1, \quad \text{and} \\ \wp(x + P_1|2P_1, 2P_2)|_{q\to 0} &= \frac{\pi^2}{4P_1^2} \sec^2\left(\frac{\pi x}{2P_1}\right) + C_2, \end{aligned} \tag{25}$$

where C_1 C_2 are constants, which will henceforth be ignored as they signify a trivial constant shift in the spectrum of the Hamiltonian. Putting $P_1 = L$ in Eq.(25) and using the result in Eq.(24) we get the trigonometric Hamiltonian. On the other hand, replacing P_1 by iL we get

$$\begin{aligned} \wp(x|2iL, 2P_2)|_{q\to 0} &= \frac{\pi^2}{4L^2}\,\mathrm{csch}^2\left(\frac{\pi x}{2L}\right), \quad \text{and} \\ \wp(x+iL|2iL, 2P_2)|_{q\to 0} &= -\frac{\pi^2}{4L^2}\,\mathrm{sech}^2\left(\frac{\pi x}{2L}\right). \end{aligned} \tag{26}$$

Changing the variable $(\pi/2L)x_j \to x_j$ we get the hyperbolic extension of the periodic Hamiltonian as

$$H_N^{h^-} = \sum_{j=1}^{N} \partial_j{}^2 - \lambda(\lambda-1)\left(\sum_{j,k}{}' \frac{1}{\sinh^2(x_{jk})} - \sum_{j,k}{}' \frac{1}{\cosh^2(x_{jk})}\right). \tag{27}$$

Following calculation similar to that in the trigonometric anti-periodic case (Sec.II B), the hyperbolic Hamiltonian in the anti-periodic model can be written in the following form,

$$H_N^{h^+} = \sum_{j=1}^{N} \partial_j{}^2 - \lambda(\lambda-1)\left(\sum_{j,k}{}' \frac{1}{\sinh^2(x_{jk})} + \sum_{j,k}{}' \frac{1}{\cosh^2(x_{jk})}\right). \tag{28}$$

The above two Hamiltonians can be written in the following compact form

$$H_N^{h^\mp} = \sum_{j=1}^{N} \partial_j{}^2 - \lambda(\lambda-1)\sum_{j,k}{}' [V(x_{jk}) \mp R(x_{jk})], \tag{29}$$

where the minus sign corresponds to the periodic case and the plus sign to the anti-periodic case.

B. Gauge momentum operators for the hyperbolic extensions

Using $v(x)$ and $r(x)$ as defined in Eq.(13), we can construct the momentum operators for the hyperbolic models. One may easily verify that $r(x)\cdot v(x) = 1$. The gauge momentum operators must satisfy the following properties to ensure the integrability of the hyperbolic models.

$$[\Lambda_{ij}, d_k] = 0. \tag{30}$$

$$\sum_{j=1}^{N} d_j^2 = H_N^{h^\mp} + \text{constant}. \tag{31}$$

$$[d_j, d_k] = 0. \tag{32}$$

$$[d_j, H_N^{h\mp}] = 0. \tag{33}$$

The above properties in turn imply that $v(x)$ and $r(x)$ be such that

$$\frac{d}{dx}v(x) = V(x)\ , \qquad v^2(x) = V(x) + \text{constant}$$
$$\frac{d}{dx}r(x) = -R(x)\ , \qquad r^2(x) = -R(x) + \text{constant}$$

From the above, we may write the momentum operators for the hyperbolic models as

$$d_j = \partial_j - \mu_1(\lambda) \sum_{\substack{k \\ k\neq j}} \coth(x_{jk})\Lambda_{jk} - \mu_2(\lambda) \sum_{\substack{k \\ k\neq j}} \tanh(x_{jk})\Lambda_{jk}. \tag{34}$$

The dependence of $\mu_1(\lambda)$ and $\mu_2(\lambda)$ on λ and the restrictions imposed on the values of λ is similar to that obtained for trigonometric models.

IV. CONCLUSION

In this article, we have studied the classical Calogero system with anti-periodic boundary condition. This system is equivalent to a Calogero system with periodic boundary condition in the presence of a transverse magnetic field. Though the presence of a magnetic field makes the system physically interesting, the anti-periodic Calogero systems are not widely studied. The reason perhaps lies in the fact that certain algebraic symmetries based on the root systems present in the periodic case are not available in the anti-periodic case. This makes the anti-periodic CSM more difficult and involved.

In this article, the integrability of the Calogero system with anti-periodic boundary condition has been established by constructing a family of commuting momentum operators. We have shown that, for certain restrictions on the values of the interaction parameter λ, the momentum operators are hermitian. Relevant extensions of the system, both classical and spin cases, should be an interesting subject of future study for investigating integrability and solvability.

V. APPENDIX

The commutation property of the gauge momentum operators is explicitly demonstrated in this appendix.

The general form of a gauge momentum operator is,

$$d_j = \partial_j + \mu_1 \sum_{\substack{m \\ m\neq j}} X_{jm}\Lambda_{jm} + \mu_2 \sum_{\substack{n \\ n\neq j}} Y_{jn}\Lambda_{jn}. \tag{35}$$

Here, X and Y are odd trigonometric or hyperbolic functions and hence their derivatives, X' and Y' are even functions.

Let us consider the successive action of d_k and d_j on ψ (where $j \neq k$),

$$\begin{aligned} d_j d_k \psi &= [\partial_j + \mu_1 \sum_{\substack{m \\ m\neq j}} X_{jm}\Lambda_{jm} + \mu_2 \sum_{\substack{n \\ n\neq j}} Y_{jn}\Lambda_{jn}][\partial_k\psi + \mu_1 \sum_{\substack{r \\ r\neq k}} X_{kr}\psi + \mu_2 \sum_{\substack{s \\ s\neq k}} Y_{ks}\psi] \\ &= \partial_j\partial_k\psi + \mu_1 \sum_{\substack{r \\ r\neq k}} X_{kr}\partial_j\psi + \mu_1 X'_{kj}\psi + \mu_2 \sum_{\substack{s \\ s\neq k}} Y_{ks}\partial_j\psi + \mu_2 Y'_{kj}\psi + \mu_1 X_{jk}\partial_j\psi \\ &+\mu_1 \sum_{\substack{m \\ m\neq\{j,k\}}} X_{jm}\partial_k\psi + \mu_1^2 \sum_{\substack{m \\ m\neq j}} X_{jm}\Lambda_{jm} \sum_{\substack{r \\ r\neq k}} X_{kr}\psi + \mu_1\mu_2 \sum_{\substack{n \\ n\neq j}} X_{jm}\Lambda_{jm} \sum_{\substack{s \\ s\neq k}} Y_{ks}\psi + \mu_2 Y_{jk}\partial_j\psi \\ &+\mu_2 \sum_{\substack{n \\ n\neq\{j,k\}}} Y_{jn}\partial_k\psi + \mu_1\mu_2 \sum_{\substack{n \\ n\neq j}} Y_{jn}\Lambda_{jn} \sum_{\substack{r \\ r\neq k}} X_{kr}\psi + \mu_2^2 \sum_{\substack{n \\ n\neq j}} Y_{jn}\Lambda_{jn} \sum_{\substack{s \\ s\neq k}} Y_{ks}\psi. \end{aligned} \tag{36}$$

Now consider the successive action of d_j and d_k on ψ

$$\begin{aligned} d_k d_j \psi &= \partial_k\partial_j\psi + \mu_1 \sum_{\substack{m \\ m\neq j}} X_{jm}\partial_k\psi + \mu_1 X'_{jk}\psi + \mu_2 \sum_{\substack{n \\ n\neq j\neq k}} Y_{jn}\partial_k\psi + \mu_2 Y'_{jk}\psi \\ &+\mu_1 X_{kj}\partial_k\psi + \mu_1 \sum_{\substack{r \\ r\neq\{k,j\}}} X_{kr}\partial_j\psi + \mu_1^2 \sum_{\substack{r \\ r\neq k}} X_{kr}\Lambda_{kr} \sum_{\substack{m \\ m\neq j}} X_{jm}\psi + \mu_1\mu_2 \sum_{\substack{r \\ r\neq k}} X_{kr}\Lambda_{kr} \sum_{\substack{n \\ n\neq j}} Y_{jn}\psi \\ &+\mu_2 Y_{kj}\partial_k\psi + \mu_2 \sum_{\substack{s \\ s\neq\{k,j\}}} Y_{ks}\partial_j\psi + \mu_1\mu_2 \sum_{\substack{s \\ s\neq k}} Y_{ks}\Lambda_{ks} \sum_{\substack{m \\ m\neq j}} X_{jm}\psi \\ &+\mu_2^2 \sum_{\substack{s \\ s\neq k}} Y_{ks}\Lambda_{ks} \sum_{\substack{n \\ n\neq j}} Y_{jn}\psi. \end{aligned} \tag{37}$$

In the commutator the first, third and fifth terms of Eq.(36) cancel the corresponding terms of Eq.(37). Of the remaining terms, collecting the terms containing the first order derivative of ψ in Eq.(36) and Eq.(37), we may write

$$\begin{aligned} &\mu_1 X_{kj}\partial_j\psi + \mu_1 \sum_{\substack{r \\ r\neq\{k,j\}}} X_{kr}\partial_j\psi + \mu_2 Y_{kj}\partial_j\psi + \mu_2 \sum_{\substack{s \\ s\neq\{k,j\}}} Y_{ks}\partial_j\psi \\ &+\mu_1 X_{jk}\partial_j\psi + \mu_1 \sum_{\substack{m \\ m\neq\{k,j\}}} X_{jm}\partial_k\psi + \mu_2 Y_{jk}\partial_j\psi + \mu_2 \sum_{\substack{n \\ n\neq\{k,j\}}} Y_{jn}\partial_k\psi \end{aligned} \tag{38}$$

$$\mu_1 X_{jk}\partial_k\psi + \mu_2 \sum_{\substack{m \\ m\neq\{k,j\}}} X_{jm}\partial_k\psi + \mu_2 Y_{jk}\psi + \mu_2 \sum_{\substack{n \\ n\neq\{k,j\}}} Y_{jn}\partial_k\psi$$
$$+\mu_1 X_{kj}\partial_k\psi + \mu_1 \sum_{\substack{r \\ r\neq\{k,j\}}} X_{kr}\partial_j\psi + \mu_2 Y_{kj}\partial_k\psi + \mu_2 \sum_{\substack{s \\ s\neq\{k,j\}}} Y_{ks}\partial_j\psi \tag{39}$$

In the above two expressions the first and the fifth terms cancel each other as X is an odd function. The same holds true for the third and seventh terms.

Finally the remaining terms in expression (38) i.e., the second, fourth, sixth and eighth cancel with the sixth, second, eighth and fourth terms respectively of expression (39) upon commutation.

Now we are left with the terms containing μ_1^2, μ_2^2 and $\mu_1\mu_2$ in the expression of the commutator. The coefficient of μ_1^2 is given by

$$\sum_{\substack{m \\ m\neq j}} X_{jm}\Lambda_{jm} \sum_{\substack{r \\ r\neq k}} X_{kr}\psi - \sum_{\substack{r \\ r\neq k}} X_{kr}\Lambda_{kr} \sum_{\substack{m \\ m\neq j}} X_{jm}\psi.$$

This term vanishes because of the symmetry in the indices j and k. By the same argument the coefficients of μ_2^2 and $\mu_1\mu_2$ also vanish. Hence it is proved that the gauge momentum operators d_j, d_k mutually commute for $j, k = 1, \ldots N$.

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[1] F. D. M. Haldane, J. Phys C 14 (1981) 2585.

[2] F. D. M. Haldane, Phys. Rev. Lett. 47 (1981) 1840; Erratum: F. D. Haldane, Phys. Rev. Lett. 48 (1982) 569.

[3] A. Gorsky and N. Nekrasov, Nucl. Phys B 414 (1994) 213.

[4] J. A. Minahan and A. P. Polychronakos, Phys. Lett. B 326 (1994) 288.

[5] A. P. Polychronakos, Phys. Rev. Lett. 74 (1995) 5153.

[6] F. J. Dyson, J. Math. Phys. 3 (1962) 140, 157, 166.

[7] H. Jack, Proc. Roy. Soc. (Edinburgh) A 69 (1970) 1.

[8] G. W. Gibbons and P. K. Townsend, Phys. Lett. B 454 (1999) 187.

[9] F. Calogero, J. Math. Phys. 10 (1969) 2191; 10 (1969) 2197.

[10] B. Sutherland, J. Math. Phys. 12 (1971) 246; 12 (1971) 251.

[11] A. P. Polychronakos, Phys. Rev. Lett. 69 (1992) 703.

[12] F. D. M. Haldane, Phys. Rev. Lett. 67 (1991) 937.

[13] Z. N. C. Ha, Phys. Rev. Lett. 73 (1994) 1574; Erratum: Z. N. Ha, Phys. Rev. Lett 74 (1995) 620.

[14] A. P. Polychronakos, Nucl. Phys. B 324 (1989) 597.

[15] Whittaker and Watson, A Course on Modern Analysis, 4th edition, Cambridge 1927.

[16] M. A. Olshanetsky, A. M. Perelomov, Phys. Rep. 94 (1983) 313.

[17] F. Calogero, Lett. Nuovo Cim. 13 (1975) 411.

[18] A. M. Perelomov, Lett. Math. Phys. 1 (1977) 531.

[19] I. M. Krichever, Funct. Anal. Appl. 14 (1980) 282.

[20] D. Bernard, M. Gaudin, F. D. M. Haldane, V. Pasquier, J. Phys. A 26 (1993) 5219, hep-th/9301084 v1.

[21] M. Cadoni, P. Carta, D. Klemm, Phys. Lett. B 503 (2001) 205.

[22] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333.

[23] J. A. Minahan, A. P. Polychronakos, Phys. Lett. B 302 (1993) 265.

[24] A. P. Polychronakos, Nucl. Phys. B 543 (1999) 485.

[25] K. Hikami, M. Wadati, J. Phys. Soc. Japan 62 (1993) 469.

[26] Z. N. C. Ha, 'Quantum Many-Body Systems in One Dimension', World Sc., Singapore 1996.

[27] E. T. Copson, 'An Introduction to the theory of functions of a complex variable', Oxford University Press, (1988).